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Phase transitions of some fully frustrated models

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Abstract. Fully frustrated antiferromagnets with triangular plaquettes are investigated. For the two-dimensional triangular lattice, the Ising antiferromagnet is in the class of the XY model with transition temperature $T = 0$. For $d \geqslant 3$ we investigate the face-centred cubic lattice. For even d there is a proper phase transition with long-range order. For odd d the fluctuation spectrum has a reduced dimensionality $(d-1)$. This is reflected in an infinite ground state degeneracy and a reduced dimensionality of the ground state order $(d-n)$. For $d = 3$ a phase transition occurs for the Ising model but not for $n \geqslant 3$. The relationship of the fully frustrated phase to the disordered spin glass is also discussed. It is suggested that the two phases might be distinct near $d = 6$ but coalesce at some lower dimensionality.

1. Introduction

The purpose of this paper is to study the phase diagram of several fully frustrated (FF) spin models. It is thus closely related to the work of Villain (1977) on the FF square and diamond lattices. Our approach differs in two respects. (i) We investigate lattices for which the elementary nearest-neighbour (NN) plaquettes are triangles, namely the two-dimensional triangular lattice and the face-centred cubic (FCC) lattice in d dimensions ($d \geqslant 3$). On such lattices the antiferromagnet is fully frustrated. This considerably simplifies the analysis because the symmetry of the underlying Bravais lattice is not broken by the introduction of frustration. For Edwards–Anderson (EA, Edwards and Anderson 1975) spin glass (SG) models with distributions of positive and negative NN spin coupling constants, one also passes continuously from ferromagnetic to FF models as the average coupling constant ($\langle J \rangle$) is changed. The FF models on these lattices are thus simply related to SG models. When the elementary plaquettes are square, the maximum EA model frustration is 50%, so that the relationship is not obvious. (ii) We also use the Landau expansion of the free energy to obtain information on the universality class and effective dimensionality of the FF phases and phase transitions.

The motivation for this investigation is largely the hope that a better understanding of FF systems may help in understanding spin glasses, and in particular the role of frustration in SG transitions (Toulouse 1977). The FF models seem to display all the essential ground state properties investigated by Kirckpatrick (1978) and by Vannimenus and Toulouse (1978). This raises the question of the existence of a separate, randomly frustrated SG phase, distinct from the FF phase, and of the properties of the relevant phase boundary. Mean field calculations suggest (T Lubensky 1978, private

§ This work was done while this author was a visitor at the Department of Physics at UCLA, the Department of Physics, University of Pennsylvania, and at the Brown Boveri Research Center.

communication) that there are indeed two separate phases. It is, however, not obvious how reliable this is for the low dimensionalities of physical interest.

In § 2 we discuss the antiferromagnet on the two-dimensional triangular lattice. The Ising model free energy was calculated exactly by Wannier (1950). We show that this model belongs in the universality class of the planar (XY) model with a sixfold symmetry-breaking term. The ground state degeneracy arises from peculiarities of the constraints on the magnitude of the spins, which also suppress the transition to $T = 0$. The degeneracy disappears for planar or Heisenberg spins. Villain's results on the FF square lattice may be understood in the same way.

In § 3 we discuss the FCC lattice for integral dimensionalities $d \geqslant 3$. For odd d we find a reduced effective dimensionality in the free energy expansions ($d_{\text{eff}} = d - 1$). For the Ising model we find an infinite one-dimensional ground state degeneracy ($\approx 2^{N^{1/d}}$), $(d-1)$-dimensional long-range order (LRO), and no zero-point entropy. In three dimensions there seems to be an effectively two-dimensional phase transition in the class of the two-dimensional Heisenberg model with cubic anisotropies.

For n-component spins with $n \geqslant d$ there is a finite ground state entropy and no phase transition for $d \leqslant 3$. The ($n = 2$) XY model is marginal. The ground state has one-dimensional order but also no entropy (degeneracy $\leqslant \lambda^{N^{2/d}}$).

In § 4 we discuss the relevance of the FF models to spin glasses.

2. Two-dimensional models

We shall mainly be concerned with the two-dimensional antiferromagnet on the triangular lattice. We note, however, that the results for the FF square lattice (Villain 1977) are very similar, and discuss this briefly below.

Wannier (1950) has calculated the thermodynamic functions and discussed the properties of the Ising model ground state in detail. He finds no specific heat singularity and a finite large entropy for the ground state. The magnetic susceptibility diverges at $T = 0$. More recently, Schick *et al* (1976, 1977) have studied the phase diagram in external fields, mainly because of its relevance to lattice gases. They predict a phase transition in any finite field (or at fixed magnetisation). Related calculations at fixed magnetisation were recently done by Berker *et al* (1978).

The Hamiltonian is

$$H = \frac{J}{2} \sum_{i,j}^{nn} \boldsymbol{S}_i \boldsymbol{S}_j = \sum_{\boldsymbol{q}} J_{\boldsymbol{q}} \boldsymbol{S}_{\boldsymbol{q}} \boldsymbol{S}_{-\boldsymbol{q}}, \tag{1}$$

where $\boldsymbol{S}_q$ is the Fourier transform of the $\boldsymbol{S}_i$, and $\boldsymbol{q}$ is a vector in the hexagonal Brillouin zone of this lattice. The minima of

$$J_{\boldsymbol{q}} = J[\cos q_x + 2\cos(q_x/2)\cos(\sqrt{3}q_y/2)] \tag{2}$$

are at the corners of the zone (Alexander 1976)

$$\boldsymbol{Q} = (\pm 4\pi/3; 0). \tag{3}$$

The other corners are related to these by reciprocal lattice vectors. The model is specified by equation (1) supplemented by the constraints

$$\boldsymbol{S}_i^2 = 1. \tag{4}$$

We first consider the ground state properties.

2.1. *Ground state properties*

For the Ising model (but not for $n \geqslant 2$) the constraints (equation (4)) are incompatible with the preferred wavevector ($\boldsymbol{Q}$). This leads to the ground state degeneracy.

Substituting $\boldsymbol{Q}$ (equation (3)) in equation (2) one finds a minimum energy of $-3J/2$ per site. On the other hand, by inspection (Wannier 1950), the lowest energy compatible with the constraints (equation (4)) is $-J$. This is the energy at the saddle point ($\pi(1, \sqrt{3}/2)$) on the zone boundary, where the constraints are automatically satisfied. Any configuration

$$\sigma(\boldsymbol{R}) = \sum_q a_q \sigma_q, \qquad a_q = a_{-q} \tag{5}$$

which satisfies the constraints and for which

$$\sum_i |a_{q_i}|^2 J_{q_i} = 1 \tag{6}$$

is also a legitimate ground state. This requires that a sufficient fraction of the contributing $\boldsymbol{q}_i$ should have $J_q < -1$. There are many ways of doing this. Wannier finds a ground state entropy of

$$S(0)/R = \frac{3}{\pi}\int_0^{\pi/6} \ln(2\cos\omega) = \mathrm{d}\omega = 0{\cdot}338314. \tag{7}$$

This essentially fortuitous origin of the Ising model ground state degeneracy becomes obvious when one considers vector models with $n \geqslant 2$. Equation (2) still holds, but solutions at Q now exist. These are (for $n = 2$) the spirals

$$(S_x + \mathrm{i}S_y) = \exp[\pm\mathrm{i}(\boldsymbol{QR} + \phi)]. \tag{8}$$

There is thus a continuous degeneracy associated with the phase (ϕ), as for the XY model. In addition there is a discrete twofold degeneracy because the spirals can be right- or left-handed.

Villain (1977) finds exactly analogous results for the FF square lattice. The only real difference is that the two degenerate points of J_q (on the square lattice) are not simply related by the lattice symmetry. This makes the interpretation somewhat more cumbersome. Obviously this reflects the fact that no realisation of the FF square lattice has the full space group symmetry.

We note that the Ising model ground state displays all the properties investigated recently by Vannimenus and Toulouse (1978), Kirckpatrick (1978) and Reed *et al* (1978) for random two-dimensional SG models. We shall show later that the ground state degeneracy for three-dimensional models has a completely different origin and remains for all n.

2.2. *Landau expansion and thermodynamics*

The structure of the free energy expansion is determined by the energy (equation (2)). For simplicity we only write down the expressions for the Ising model. The generalisation to n-component vectors is straightforward, but keeping track of the n-component vector coupling complicates the notations without adding anything to the argument. We define

$$\eta_k = \sigma_{Q+k} = \eta^{+}_{-k}, \tag{9}$$

where $\boldsymbol{k}$ is confined to a triangle centred at $\boldsymbol{Q}$. The order parameter is complex:

$$\eta = \eta_0 = \sigma_{\boldsymbol{Q}} = |\eta|\, e^{i\phi}, \qquad \eta^+ = \sigma_{-\boldsymbol{Q}}. \tag{10}$$

The phase ϕ relates the spin density wave (η) to the underlying lattice. The free energy functional can be written

$$\Phi = \Phi_2 + \Phi_4 + \Phi_6 + \dots . \tag{11}$$

The structure of Φ_2 is given by $J_{\boldsymbol{q}}$. The leading order (in k) we write

$$\Phi_2 = \int \mathrm{d}\boldsymbol{k}\, (r + k^2)(\eta_{\boldsymbol{k}} \eta^+_{-\boldsymbol{k}}). \tag{12}$$

For small k the only contribution to Φ_4 is

$$\Phi_4^1 \approx \int \prod_{i=1}^{4} (\mathrm{d}\boldsymbol{k}_i) \delta\Big(\sum_{i=1}^{4} \boldsymbol{k}_i\Big) \eta_{\boldsymbol{k}_1} \eta_{\boldsymbol{k}_2} \eta^+_{\boldsymbol{k}_3} \eta^+_{\boldsymbol{k}_4}, \tag{13}$$

which does not depend on the phase ϕ. Other terms can occur when

$$\Big|\sum_i \boldsymbol{k}_i\Big| \geqslant Q. \tag{14}$$

The most important is

$$\Phi_4^2 \approx \sigma_0 \int \prod_{i=1}^{3} (\mathrm{d}\boldsymbol{k}_i) \delta\Big(\sum_{i=1}^{3} \boldsymbol{k}_i\Big) (\eta_{\boldsymbol{k}_1} \eta_{\boldsymbol{k}_2} \eta_{\boldsymbol{k}_3} + \text{c.c.}), \tag{15}$$

where we have used the fact that

$$3\boldsymbol{Q} = \boldsymbol{K} \tag{16}$$

is a reciprocal lattice vector. This term becomes important in the presence of a uniform magnetic field (H), or for a lattice gas, when $\sigma_0 \neq 0$. It leads to a cos 3ϕ, three-state Potts-model-like dependence on ϕ (Alexander 1976).

In Φ_6 there are also independent contributions analogous to Φ_4 (equation (13)) with products of three $\eta_{\boldsymbol{k}_i}$ and three $\eta^+_{\boldsymbol{k}_i}$. More important is the fact that the rotational symmetry in ϕ is broken:

$$\Phi_6^2 \sim \int \prod_{i=1}^{6} (\mathrm{d}\boldsymbol{k}_i) \delta\Big(\sum_{i=1}^{6} \boldsymbol{k}_i\Big) \Big(\prod_{i=1}^{6} \eta_{\boldsymbol{k}_i} + \text{c.c.}\Big). \tag{17}$$

When there are no external fields, this is the first symmetry-breaking term. It implies a cos6ϕ symmetry-breaking term.

One thus has a model in the universality class of the XY($n = 2$) model with a sixfold symmetry-breaking term. From the results of José *et al* (1977) one would thus expect two phase transitions: an XY-like transition at T_{XY}; and, since cos6ϕ is irrelevant at T_{XY}, a locking transition at some lower temperature. This does not seem to agree with the thermodynamic results of Wannier (1950), which show no singularities. Additional information can be obtained from the magnetic susceptibility, which is finite at all finite temperatures, but diverges at $T = 0$. As pointed out above (equation (15)), a uniform magnetic field induces a cos3ϕ symmetry-breaking term. (Note that this is not the field conjugate to the order parameter η.) Moreover, such a term would be relevant at T_{XY} (José *et al* 1977), which implies an infinite susceptibility at and below T_{XY}. Thus $T_{\mathrm{XY}} > 0$ is certainly excluded. On the other hand, Shick *et al* (1976, 1977) find a finite transition temperature for all finite fields. This, together with the susceptibility divergence, strongly suggests $T_{XY} = 0$.

The FF and NN model is also a special point on the phase diagram in another way. The transition occurs at a finite temperature as soon as next-nearest-neighbour (NNN) interactions are added (see e.g. Schick *et al* 1977, Mihura and Landau 1977). One notes that this only shows up in Φ by a quantitative change in $J_q(\Phi_2)$.

It is interesting to generalise to n-component spins. We have seen in § 2.1 that the ground state degeneracy disappears. For $n = 2$ the order parameter has a free phase (equation (8)) and an additional twofold (right–left) degeneracy. This would give an (n, m) type of order parameter (see e.g. Aharoni 1976) with $n = m = 2$. Since the two types of spirals are coupled by the constraints, there should be hypercubic couplings in one (say the m) spin substance. Villian (1977) finds the same structure on the square lattice. While such models have been studied (Domany and Riedel 1978), it is not even clear if a phase transition is predicted. For $n \geq 3$ there is certainly no transition.

3. Antiferromagnetic face-centred cubic lattices

We define a d-dimensional FCC lattice for $d \geq 2$ by straightforward generalisation from three dimensions. The lattice sites are given by

$$\boldsymbol{R}(n_1) = \tfrac{1}{2}a(n_1, n_2 \ldots n_d), \qquad \sum n_i = 2m, \tag{18}$$

where the n_i and m are (positive or negative) integers. The NN separations are given by vectors of the type

$$\boldsymbol{r}_{\mathrm{nn}/a} = (\pm\tfrac{1}{2}; \pm\tfrac{1}{2}, 00 \ldots 0), \qquad r^2_{\mathrm{nn}/a^2} = \tfrac{1}{2}. \tag{19}$$

Thus each site has

$$\nu_d = 2d(d-1) \tag{20}$$

nearest neighbours, and each of these has $4(d-2)$ nearest neighbours in the original NN shell. Altogether there are

$$\mu_d = 4d(d-1)(d-2) \tag{21}$$

elementary triangular NN plaquettes associated with each lattice point.

The reciprocal lattice is body-centred cubic and is generated by the vectors

$$K = 2K_0(l_i \ldots l_d), \qquad K_b = K_0(1, 1 \ldots 1), \tag{22}$$

where the l_i are integers and

$$K_0 = 2\pi/a. \tag{23}$$

We consider the NN antiferromagnet on this lattice. All elementary plaquettes are frustrated. For the Ising model the ground state energy cannot be lower than $-J/3$ per bond. This assumes that all elementary plaquettes are in their minimum-energy configurations. This gives a lower bound

$$U/J > -d(d-1)/3 \tag{24}$$

for the ground state energy per site.

It is easy to see that

$$J_{\boldsymbol{q}}=-2J\sum_{\alpha<\beta}^{d}\cos(q_\alpha a/2)\cos(q_\beta a/2). \tag{25}$$

There is always a ferromagnetic minimum at the origin,

$$J_0/J=-d(d-1). \tag{26}$$

We are interested in the maximum of $J_{\boldsymbol{q}}$ (for positive J). For odd d the maximum is degenerate and occurs on lines on the surface of the Brillouin zone. On these lines one has

$$J_{\boldsymbol{q}}/J=d-1. \tag{27}$$

This degeneracy is well known in NN tight-binding or lattice dynamics calculations. In three dimensions it occurs on the line

$$\boldsymbol{q}=K_0(0,1,z), \tag{28}$$

and on the lines obtained by permutations of the coordinates. These lines form crosses on the square faces of the Brillouin zone. For general odd d a representative line is

$$\boldsymbol{q}=K_0(l_1,\dots l_{d-1},x_d), \qquad \begin{array}{ll} l_i=0, & 1\leqslant i\leqslant (d-1)/2 \\ l_i=1, & (d+1)/2\leqslant i\leqslant d-1. \end{array} \tag{29}$$

There are in general, by symmetry,

$$n_d^0\times\frac{d}{2}\binom{d-1}{(d-1)/2} \tag{30}$$

independent lines of this type.

For even d the maximum occurs at discrete points on the zone boundary of the type

$$Q=K_0(l,\dots l_d), \qquad \begin{array}{ll} l_i=0, & 1\leqslant i\leqslant d/2 \\ l_i=1, & d/2+1\leqslant i\leqslant d, \end{array} \tag{31}$$

and there are

$$n_d^e=\frac{1}{2}\binom{d}{d/2} \tag{32}$$

points of this type. At these points

$$J_q/J=d. \tag{33}$$

3.1. Ground state properties

Consider first the situation for d even. The spin density waves

$$\sigma_{\boldsymbol{Q}}(\boldsymbol{R}_i)\approx\cos(\boldsymbol{Q}\boldsymbol{R}_i) \tag{34}$$

have amplitudes ± 1 at all lattice sites, where $\boldsymbol{R}_i$ is given by equation (18) and $\boldsymbol{Q}$ by equation (31). They are thus compatible with the constraints (equation (4)). For the Ising model there are no other minimum-energy solutions, and the ground state degeneracy is therefore $2n_d^e$ (equation (32)). For $n\geqslant 2$ the continuous degeneracy is added.

In all cases the ground state has LRO and the frustrations seem to have no important effects. We note that for $d=4$ the minimum energy (equation (33)) coincides with the frustration estimate (equation (24)). For $d>4$ (and even) $(d-1)/3>1$, and it is no longer possible to construct configurations for which all elementary triangles have their minimum-energy configuration.

For odd dimensionality the ground state is infinitely degenerate and has, at most, $(d-1)$-dimensional LRO. To see this consider a $(d-1)$-dimensional lattice orthogonal to any of the cubic axes. It constitutes a $(d-1)$-dimensional FCC lattice $(d-1)$ even), and one can therefore construct a minimum-energy state of the type (31) for the internal interactions of these layers. There are $2n^{e}_{d-1}$ (equation (32)) ways of doing this, and the energy per spin is (equation (33))

$$|U/J| = -(d-1). \tag{35}$$

This is also the minimum energy attainable in d dimensions (equation (27)). Thus successive $(d-1)$-dimensional layers do not interact. For each cubic axis there are $(2n^{e}_{d-1})^{N^{1/d}}$ different ways of stacking the spin configurations.

For the Ising model, configurations with different stacking axes are incompatible, and the total ground state degeneracy is

$$D_{\mathrm{I}} = d\left(\binom{d-1}{(d-1)/2}\right)^{N^{1/d}}. \tag{36}$$

Thus the ground state is infinitely degenerate but has no finite entropy. All ground state configurations have $(d-1)$-dimensional LRO.

For higher spin dimensionality (n) the degeneracy is higher. We only set a lower limit. The Ising model structures have a constant amplitude at all lattice sites. If one chooses one of these configurations, with amplitude u_α for each of the n spin components ($\Sigma_{\alpha=1}^{n} u_\alpha^2 = 1$), one always obtains a possible ground state configuration for the n-component spin. Thus from equation (36)

$$\lg D_n \begin{cases} \geqslant N^{n/d}, & n \leqslant d \\ \approx N, & n \geqslant d, \end{cases} \tag{37}$$

and the ground state entropy must become finite for $n \geqslant d$. Other structures (e.g. spirals) are certainly possible, so that equation (37) only sets a lower limit on the degeneracy.

In three dimensions the layers are simply antiferromagnetic square lattices, and it is easy to see by inspection that they do not interact. We note that for $n<d$ the ground states we have constructed have $(d-n)$-dimensional LRO.

We do not know if this alternation between even and odd dimensionalities is a general phenomenon or a peculiarity of the FCC lattice we consider. We note that Villain (1967) finds the same type of layered degeneracy for the diamond lattice, and the results of Kirckpatrick (1977) are also consistent with this structure of the ground state.

We also note that the degeneracy increases smoothly as n increases, and the anomalies we encountered for the two-dimensional case do not occur.

3.2. *Phase transitions*

3.2.1. General structure of the continuous models. The order parameter is a spin density wave. For even dimensions it has n^{e}_{d} components representing the degeneracy of the points $\boldsymbol{Q}$ and n spin components. It is thus of the (n, n_d^e) type with hypercubic terms resulting from the constraints. Thus one predicts a fairly standard type of transition. The only effect of the frustration is to change the universality class.

For odd d the situation is quite different. The order parameter is associated with the lines $\boldsymbol{q}$ (equation (29)). These form a net on the surface of the Brillouin zone with intersections of $(d+1)/2$ lines at two points on each line. Except for small corrections one can therefore regard each line as an independent component of the order parameter. Thus one has:

(*a*) The order parameter has three indices—an index $\alpha (1 \leq \alpha \leq n)$ for the spin component, an index $\mu (1 \leq \mu \leq n_d^o)$ for the line, and a continuous index to describe the position along the line.

(*b*) The propagator has an effective dimensionality $d-1$. It involves an expansion around a line rather than a point.

(*c*) The fourth-order terms include hypercubic terms coupling the different lines (μ). They also have an explicit dependence on the continuous index.

Thus one has a model with a reduced effective dimensionality $(d-1)$ and rather complex order parameter and fourth-order terms. In most cases a reduced effective dimensionality implies a first-order transition with a crossover to the full dimensionality of the ordered state (Brazovski 1975). For the present model the reduced dimensionality seems to persist to the ground state, so that the crossover may not occur. Each of the continuous models one obtains can of course be studied formally for arbitrary dimensionality. Explicitly, all have an (upper) critical dimensionality of four (for the fluctuation spectrum). Thus for $d \geq 4$ the FF FCC models should have mean field-like transitions in spite of the infinite degeneracy of the ground state.

The only really interesting case is $d = 3$. Since the effective dimensionality is two, the behaviour of the model depends in an essential way on the spin dimensionality and on the structure of the fourth- and higher-order terms in the free energy functional.

3.2.2. The three-dimensional Ising model. The order parameter is associated with the lines $\kappa_0(0, 1, \lambda)$ (equation (28)). It can be seen by expansion of $J_{\boldsymbol{q}}$ (equation (25)) around these lines that the low-energy density of states is indeed constant, and the fluctuation spectrum is therefore two-dimensional. On the lines we can define a three-component order parameter

$$\eta_{i\zeta} = \sigma(\boldsymbol{q}_{i\zeta}), \qquad -1 \leq \zeta \leq 1, \tag{38}$$

where

$$\boldsymbol{q}_{1\zeta} = \kappa_0(\zeta, 1, 0), \qquad \boldsymbol{q}_{2\zeta} = \kappa_0(0, \zeta, 1), \qquad \boldsymbol{q}_{3\zeta} = \kappa_0(1, 0, \zeta),$$

leading to a contribution

$$r \sum_i (n_i)^2 = r \sum_i \int \mathrm{d}\zeta \, \eta_{i\zeta} \eta_{i-\zeta} \tag{39}$$

to Φ_2. Also, as we have seen in discussing the ground state, the three components are incompatible in forming ground state structures. One therefore obtains a negative cubic anistropy term

$$-v \sum_i \iiint \mathrm{d}\zeta \, \mathrm{d}\zeta' \, \mathrm{d}\zeta'' \, \eta_{i\zeta} \eta_{i,\zeta'} \eta_{i,\zeta''} \eta_{i,-\zeta-\zeta'-\zeta''}. \tag{40}$$

Thus the FF FCC lattice yields a model closely related to the two-dimensional Heisenberg ($n = 3$) model with cubic anistropy terms. This model has a phase transition at finite temperature. The analogy is, however, not complete. The three-dimensional structure of the lattice shows up in the explicit dispersion relations when one moves away from the external lines (equation (28)). Thus while the density of states

$$\rho(\epsilon) = \int \mathrm{d}\boldsymbol{q}\, \delta[J(\boldsymbol{q}) - \epsilon] \tag{41}$$

is two-dimensional ($\rho(\epsilon) \xrightarrow[\epsilon \to 0]{}$ constant), there is no simple way to define the degeneracy parameter (ζ) and the two-dimensional coordinates away from the degeneracy lines. The three-dimensional character also shows up in the topology of the defect structure. For a fixed disorder axis, changes in the stacking of the ordered planes only require two-dimensional 'Bloch' lines. Boundaries between regions with different axes almost always require discontinuities on surfaces as for three-dimensional cubic magnets. We believe that these differences are probably irrelevant and in any case should not inhibit the phase transition. The argument is, however, not conclusive, and a more careful study is certainly required.

We have also tried to investigate the thermal stability of the two-dimensional LRO in the ground state. In a low-temperature expansion the interaction between spin deviations on adjacent planes can be attractive or repulsive with equal probability. The result is that correlated pairs of excitations have a larger statistical weight than they would have for non-interacting layers. This is the effect of the competing alternative structures and will certainly lower the (Ising) transition temperature. Changes in LRO are, however, always associated with surfaces or lines of discontinuity. One therefore concludes that the order should persist to some finite temperature.

Analogous arguments for the ($n = 2$) XY model are too vague to be meaningful. It seems fairly certain that for $n > 3$ there is no phase transition.

4. Conclusions and relation to spin glasses

Most theoretical discussions of spin glasses are based on the EA model with a distribution of positive and negative spin interactions. They thus have randomly distributed frustrated plaquettes, but also gauge disorder (Toulouse 1977) and frequently a distribution in the interaction strengths.

As shown by Toulouse and discussed more recently by Fradkin *et al* (1979), gauge disorder is extremely important in interpreting experimental results but does not effect the thermodynamics.

For lattices with triangular plaquettes, the FF model and the ferromagnet occur naturally as limiting cases of an EA model. It is therefore natural to assume that there are three distinct phases: ferromagnetic for a low concentration of frustration; SG; and finally a distinct FF phase for sufficiently high density of frustration. A mean field calculation similar to that of Chen and Lubensky (1977) actually predicts this type of phase diagram (T Lubensky 1978, private communication). Also the upper critical dimensionality for SG field theories is known to be six (Harris *et al* 1976). For the FF models it is four for even models and five for odd models. Thus the SG and FF phases must be distinct, with a separating phase boundary, near six dimensions. The situation is less clear at lower dimensionalities. There is considerable disagreement in the literature as to the lower critical dimensionality for spin glasses. It is hard to see physically how the lower critical dimensionality for disordered SG (d^l_{SG}) could be higher than that for FF (d^l_{FF}). This would imply that for $d^l_{FF} < d < d^l_{SG}$ a paramagnetic phase appears on the phase diagram between the ferromagnet and FF, and an ordering transition (from the paramagnetic state) is induced by increasing the concentration of frustrations. Our results for d^l_{FF} are consistent with those of Anderson and Pond (1978) for d^l_{SG}. The phase diagram does, however, also allow a different possibility. The ferromagnetic SG phase boundary could merge with the SG–FF boundary below some critical dimensionality $\bar{d}^l_{SG}$. The distinct SG phase described by the field theories would thus disappear at $\bar{d}^l_{SG}$. On the other hand, a physical SG with random frustration would show a (FF) transition down to d^l_{FF}. If this is indeed the case, it might explain the contradiction between the results of Fisch and Harris (1977) and those of Anderson and Pond (1978).

An indication that there might be a third critical dimensionality in the problem is given by our results on the ground state of the FF FCC lattices. The minimum energy attainable is given by equations (27) and (33). This is consistent with a minimum energy arrangement on all plaquettes only for one dimensionality (d_0). Below d_0 one presumably has a ground state degeneracy of the type we saw in two dimensions for the Ising model ($d_0 = 3$ or 4). For $d > d_0$ the number of unsatisfied bonds is always *larger* than would be nominally required by the plaquettes. At least for Ising spins this is certainly an interference effect reflecting the high density of frustrations. Thus it seems that models with $d > d_0$ are somehow different from those with $d < d_0$ and might therefore have a different phase diagram.

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References

Aharoni A 1976 *Phase Transitions* vol 6, ed. C Domb and M Green (New York: Academic)
Alexander S 1976 *Phys. Lett.* **23** 491
Anderson P W and Pond C M 1978 *Phys. Rev. Lett.* **40** 903
Berker A N, Ostwald S and Putnam F A 1978 *Phys. Rev.* B **17** 3650
Brazovski S A 1975 *Sov. Phys.-JETP* **41** 85
Chen J H and Lubensky T C 1977 *Phys. Rev.* B **16** 2106
Domany E and Riedel E K 1978 *Phys. Rev. Lett.* **40** 561
Edwards S F and Anderson P W 1975 *J. Phys. F: Metal Phys.* **5** 965
Fisch R and Harris A B 1977 *Phys. Rev. Lett.* **28** 285
Fradkin E, Huberman B A and Schenker S H 1979 *Phys. Rev.* B
Harris A B, Lubensky T C and Chen J H 1976 *Phys. Rev. Lett.* **36** 415
José J V, Kadanoff L P, Kirckpatrick S and Nelson D R 1977 *Phys. Rev.* B **16** 1217
Kirckpatrick S 1977 *Phys. Rev.* B **16** 4630
Mihura B and Landau D P 1977 *Phys. Rev. Lett.* **38** 977
Reed P, Moore M A and Bray A J 1978 *J. Phys. C: Solid St. Phys.* **11** L139

Schick M, Walker J S and Wortis M 1976 *Phys. Lett.* A **58** 479
—— 1977 *Phys. Rev.* B **16** 2295
Toulouse G 1977 *Commun. Phys.* **2** 115
Wannier G H 1950 *Phys. Rev.* **78** 341
Vannimenus J and Toulouse G 1977 *J. Phys. C: Solid St. Phys.* **10** L537
Villain J 1977 *J. Phys. C: Solid St. Phys.* **10** 1717